

Global existence for a strongly coupled reaction diffusion system

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Abstract

In this work we use functional methods to prove the boundedness and global existence of solutions for a class of strongly coupled parabolic systems. We apply the results to deduce the global existence of solutions for a classic Shigesada-Kawasaki-Teramoto (SKT) type model for an extended range of the self-diffusion and cross-diffusion coefficients than those available in the current literature. We perform numerical simulations in 2D, via a spectral Galerkin method to verify our global existence results, as well as to visualize the dynamics of the system.

Keywords: strongly coupled parabolic system, SKT model, spectral Galerkin method.

1 Introduction

Modeling species interactions occupies a central theme in spatial ecology and mathematical biology. Many of these models take the form of reaction diffusion equations, where the reaction terms describe both inter-species and intra-species dynamics, including factors such as birth and death. The diffusion represents random spatial movement of the species. Recently there has been a lot of activity concerning models where the diffusion of one species may be influenced by another species. These models come under the class of strongly coupled parabolic systems [4, 24] and are also known in the literature as cross-diffusion systems. Although these models may be more realistic from a biological or modeling point of view (as they naturally incorporate inter-species and intra-species competition), they are more difficult to handle mathematically than their constant coefficient, pure diffusion counterparts. Their classical applications are immense and there are new potential applications ranging from bee pollination [41] to pedestrian flow [3]. Among the first efforts to model cross-diffusion was the model system proposed in [36] by Shigesada, Kawasaki and Teramoto. This model and its variants are now known as SKT models. For a detailed steady state analysis of this model one can see [24]. In the current work, we consider the following cross-diffusion system which is a generalized form of the SKT model,

$$\frac{\partial u}{\partial t} - \nabla [P^u \nabla u + P^v \nabla v] = f(u, v) = u(a_1 - b_1 u + c_1 v) \quad \text{in } \mathbb{R}^+ \times \Omega, \quad (1)$$

$$\frac{\partial v}{\partial t} - \nabla [Q^u \nabla u + Q^v \nabla v] = g(u, v) = v(a_2 + b_2 u - c_2 v) \quad \text{in } \mathbb{R}^+ \times \Omega, \quad (2)$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \quad (3)$$

and the initial data

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad \text{in } \Omega, \quad (4)$$

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where

$$\begin{cases} P^u = d_1 + \alpha_{11}u + \alpha_{12}v, & P^v = b_{11}u, \\ Q^u = b_{22}v, & Q^v = d_2 + \alpha_{21}u + \alpha_{22}v. \end{cases} \quad (5)$$

Also, to simplify the notations, we let

$$P = P^u \nabla u + P^v \nabla v, \quad Q = Q^u \nabla u + Q^v \nabla v.$$

The domain Ω is an open bounded domain of class \mathbb{C}^1 in \mathbb{R}^N , with boundary $\partial\Omega$ and $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$. The components u and v are the solution to system (1)-(2), are nonnegative functions, and represent the population densities of the prey and predator species, respectively. d_1 and d_2 are positive constants representing the random diffusion rates of the two species respectively. The initial data are functions in $W^{1,p}(\Omega)$ ($p > N$) and assumed to be nonnegative which imply, via the maximum principle [38], the positivity of the solution on its interval of existence. The constants a_i, b_i, c_i ($i = 1, 2$) are any real numbers with $b_1, c_2 > 0$, where b_2 and c_1 are either nonpositive or both positive, and sufficiently small. Also we may have $c_1 b_2 < 0$, with one positive and one negative, with the positive one sufficiently small in (48), (49). The constants α_{ij}, b_{ii} ($i, j = 1, 2$) are nonnegative constants: α_{11} and α_{22} are the self-diffusion rates, α_{12} and α_{21} are the cross-diffusion rates and b_{11}, b_{22} are the gradient cross-diffusions. When $\alpha_{12} = b_{11}$, $\alpha_{21} = b_{22}$ and $b_1 < 0, c_2 < 0$ then (1)-(2) is in fact the SKT model [36].

However when the reaction diffusion systems include cross-diffusion as in (1)-(2), there are not many results: When $d_1 > d_2$, $\alpha_{12} = \alpha_{22} = 0$, $a_1 = b_1 = c_2 = 0$, an analogous system describing epidemics, M. Kirane and S. Kouachi established in a series of papers [12, 13, 14] studying global existence and asymptotics. Shim [37] proved the existence of global solutions to (1)-(4) in the space dimension $N = 1$, when $d_1 = d_2$ and $\alpha_{11} = \alpha_{22} = 0$ or when $0 < \alpha_{21} < 8\alpha_{11}$ and $0 < \alpha_{12} < 8\alpha_{22}$. Also in the space dimension $N = 1$, the same author established global existence when $\alpha_{21} = 0$. Recently, Xu in [39] showed global existence when $\alpha_{12} = 0$ in two cases: when $\alpha_{11} = 0$ or when $\alpha_{11} > 0$ and $N < 10$. The condition $\alpha_{12} = 0$ implies by simple application of the maximum principle, the uniform boundedness of $u(t, \cdot)$ on $[0, T_{\max}] \times \Omega$, where T_{\max} denotes the eventual blowing-up time in $\mathbb{L}^\infty(\Omega)$. There is a large literature on this class of models, and the interested reader is referred to [19, 23, 25, 27, 28, 30, 34, 35]. The global existence of weak solutions under natural conditions is settled [5, 7]. Recently a number of sophisticated tools that include the use of Compatanio and Morrey spaces, as well as Entropy functional techniques have been devised and applied to these problems. For details the reader is referred to [4, 21, 22]. Also, there have been recent efforts to prove global boundedness of weak solutions to cross diffusion systems, using an entropy functional approach [11]. On the more applied side, Yong et al. [41] considered a diffusion system, modeling the interactions between honey bees and solitary bees using a form of the SKT model. They derive a number of interesting conclusions from their work by restricting the self-diffusion and cross-diffusion coefficients and compare their simulation results to field data [41].

We focus on [20], where the author considered (1)-(4), however he imposed the restriction that the reaction terms should be negative if either u or v is sufficiently large. This is not necessary in our case, since the second reaction term can be large for $u \gg v$. In [20] the author proves global existence when b_2 and c_1 are non-positive, under the following conditions

$$\begin{aligned} (i) & \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} - b_{11}b_{22} \geq 0, \\ (ii) & \alpha_{22} - \alpha_{12} > b_{11}, \\ (iii) & \alpha_{11} - \alpha_{21} > b_{22}. \end{aligned} \quad (6)$$

When $\alpha_{ij} = 0$ ($i, j = 1, 2$), and b_1 and c_2 are of opposite signs, the system is the well-known Lotka-Volterra prey-predator model. In this case and for more general reactions and boundary conditions many results have been obtained. For more details see [1, 9, 26, 29].

Our primary contributions in this paper are the following:

1) We prove boundedness of solutions to (1)-(4) via Theorem 2.1, when $\alpha_{11} > \alpha_{21}$ and $\alpha_{22} > \alpha_{12}$ under the following condition

$$(\alpha_{11} - \alpha_{21})(\alpha_{22} - \alpha_{12}) > b_{11} b_{22}, \quad (7)$$

It is clear that our Condition (7) is weaker than (6). We next use the everywhere regularity result of bounded solutions to (1)-(4), via Theorem 1.1 from [20] to:

2) Deduce global existence of solutions, without restriction on the space dimension, to (1)-(4), via our new Condition (10). This is more general than [20]. This is accomplished via Theorem 2.2.

We then perform numerical simulations under the following strategy:

3) We choose parameters that satisfy Condition (10), whilst violating (6). With this choice of parameters we simulate (1)-(4) for a host of different initial conditions. Here we demonstrate that there are a host of initial conditions, for which we have globally existing solutions, for which the results of [20] are not applicable, but our result Theorem 2.2 is indeed applicable.

For the benefit of the reader we recap the Hölder continuity result via Theorem 1.1, from [20]

Theorem 1.1 ([20]). *Assume that $d_i, \alpha_{ij}, b_{ii} > 0$, $i, j = 1, 2$ and*

$$\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} - b_{11}b_{22} \geq 0. \quad (8)$$

Let $V_1 = (\alpha_{11} - \alpha_{21} - b_{22})$ and $V_2 = (\alpha_{22} - \alpha_{12} - b_{11})$. We assume that

$$\begin{aligned} & (i) \ V_1 = 0 \text{ and } V_2 \neq 0, \text{ and vice versa, or} \\ & (ii) \ V_1 V_2 > 0, \text{ or} \\ & (iii) \ (d_1 - d_2)(V_2 - V_1)[(\alpha_{11} - \alpha_{21})(\alpha_{22} - \alpha_{12}) - b_{11}b_{22}] > 0. \end{aligned} \quad (9)$$

Then bounded positive weak solutions to (1)-(4) are Hölder continuous.

Remark 1.2. *This everywhere regularity result of bounded solutions to (1)-(4) obtained in [20] is very interesting. However, the author was hindered, by not being able to show the boundedness and global existence when $V_1 = 0$ or $V_2 = 0$. He was only able to prove global existence in the case when $V_1 > 0$ and $V_2 > 0$. Furthermore we remark that the second and third inequalities in Conditions (6) (which was assumed to prove global existence in [20]) imply the first inequality when it is a strict inequality, and contradict it in the case of equality.*

In Section 2 we state our main results, then we present our proofs in Section 3. Lastly, in Section 4 we present numerical simulations, that support our results Theorems 2.1, 2.2.

2 Statement of the main results

For initial conditions in $W^{1,p}$ ($p > N$), it was proved in [2] that solutions to problems more general than (1)-(4) exist locally in time. For the global existence, it was proved that solutions to (1)-(4) exist globally in time if one has control on both of their \mathbb{L}^∞ and Hölder norms. This is in contrast to classical reaction diffusion systems (see [6, 8, 32, 33, 38]) where one need only control the L^∞ norms of solutions. Counterexamples in [10] confirmed this: bounded weak solutions of certain strongly coupled systems may blow up in finite time.

Our aim is to construct a polynomial Lyapunov functional (see [15, 16, 17, 18, 20, 31]) depending on the solution (u, v) of system (1)-(2). We then use functional methods to derive their L^∞ bounds, via this functional. Next, we apply known results to show their Hölder continuity, and thus deduce their global existence. More precisely, we will use a quadratic form according to the solution (u, v) of system (1)-(4) to prove the following result concerning the boundedness of solutions

Theorem 2.1. *Consider $\alpha_{11} > \alpha_{21}$ and $\alpha_{22} > \alpha_{12}$, then under the following condition*

$$(\alpha_{11} - \alpha_{21})(\alpha_{22} - \alpha_{12}) > b_{11}b_{22},$$

positive solutions of problem (1)-(4) are bounded on $[0, T_{\max}[$ by constants depending on initial data and the reaction terms.

Then, we will use Theorem 1.1 to deduce that bounded solutions of system (1)-(2) are Hölder continuous. We will give under stronger assumptions than those in Theorem 1.1, the proof of the following

Theorem 2.2. *Assume that*

$$\begin{aligned} (i) & V_1 = 0 \text{ and } V_2 > 0, \text{ and vice versa, or} \\ (ii) & \alpha_{11} - \alpha_{21} > b_{22} \text{ and } \alpha_{22} - \alpha_{12} > b_{11}, \text{ or} \\ (iii) & d_1 > d_2, \alpha_{11} - \alpha_{21} > b_{22}, \alpha_{22} - \alpha_{12} < b_{11} \text{ and (7), or} \\ (iv) & d_1 < d_2, \alpha_{11} - \alpha_{21} < b_{22}, \alpha_{22} - \alpha_{12} > b_{11} \text{ and (7),} \end{aligned} \quad (10)$$

then weak solutions with nonnegative initial data to (1)-(4) are classical and exist globally.

Remark 2.3. *Note that our Conditions (10) are weaker than (6). For example, when $d_1 > d_2$ we can take $0 < \alpha_{11} - \alpha_{21} < b_{22}$ and $(\alpha_{22} - \alpha_{12}) > b_{11}$ such that Condition (iii) in (6) is not satisfied, and the results of [20] are not applicable. However, our Condition (iv) in (10) is still satisfied, and we have global existence.*

We consider here systems of two equations with homogeneous Neumann boundary conditions, but our main results are applicable to those of more equations and with suitable other boundary conditions.

For a given function w , we will denote by w_+ the nonnegative part of w , $\sup\{w, 0\}$ and with a slight abuse of notation, we will write $H_u = \frac{\partial H}{\partial u}(u, v)$, $H_{uv} = \frac{\partial^2 H}{\partial u \partial v}(u, v)$, $\nabla H = \nabla_x (H(t, x))$ and so on.

3 Proofs

In this section, we study the boundedness of solutions to (1)-(4), for this purpose we consider the following quadratic form

$$H(u, v) = \frac{1}{2}\lambda u^2 + uv + \frac{1}{2}\mu v^2, \quad (11)$$

where λ and μ are positive constants such that $\lambda\mu = K^2$ for some constant K which we suppose > 1 to assure the positive definiteness of $H(u, v)$. We define a subset of \mathbb{R}_+^2 as a neighborhood of a local solution as follows

$$\Gamma = \{(u(t, x), v(t, x)) : 0 < t < T_{\max}, \quad x \in \Omega\}. \quad (12)$$

For the proof of the Theorem 2.1 on the boundedness of solutions to (1)-(4), we need some Lemmas and apply the following (see D. Le [20])

Theorem 3.1. *If there exist positive real numbers λ_1 and C_0 such that*

$$\begin{cases} P \cdot \nabla H_u + Q \cdot \nabla H_v \geq 0, \\ (H_u \cdot P + H_v \cdot Q) \nabla H \geq \lambda_1 |\nabla H|^2, \end{cases} \quad (13)$$

and

$$H_u f + H_v g \leq 0, \quad (14)$$

for all $(u, v) \in \Gamma \cap \{(u, v) : H(u, v) > C_0\}$, then the solution (u, v) of Problem (1)-(4) is bounded.

Proof. We use the following functional

$$L(t) = \frac{1}{2} \int_{\Omega} [(H - C)_+]^2 dx, \quad (15)$$

where $C > \max\{C_0, H_0\}$ and $H_0 = \sup_{x \in \Omega} H(u_0(x), v_0(x))$.

Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_{\Omega} [H_u \nabla P + H_v \nabla Q] (H - C)_+ dx + \int_{\Omega} \{H_u f(u, v) + H_v g(u, v)\} (H - C)_+ dx \\ &:= I + J, \end{aligned} \quad (16)$$

where

$$I = \int_{\Omega} (H - C)_+ [H_u \nabla P + H_v \nabla Q] dx, \quad (17)$$

and

$$J = \int_{\Omega \cap \{H > C\}} (H - C)_+ \{H_u f(u, v) + H_v g(u, v)\} dx. \quad (18)$$

Then by simple application of Green's formula with the boundary conditions (3), we get

$$\begin{aligned} I &= - \int_{\Omega} \{ \nabla [H_u (H - C)_+] \cdot P + \nabla [H_v (H - C)_+] \cdot Q \} dx \\ &= - \int_{\Omega \cap \{H > C\}} \{ (H_u P + H_v Q) \cdot \nabla H + (P \cdot \nabla H_u + Q \cdot \nabla H_v) (H - C)_+ \} dx. \end{aligned} \quad (19)$$

Using (13) we get

$$I \leq -\lambda_1 \int_{\Omega \cap \{H > C\}} |\nabla H|^2 dx.$$

From (14), we have $J \leq 0$, then by integration with respect to t we get

$$\int_{\Omega} [(H - C)_+]^2 dx \Big|_0^t + \lambda_1 \int_0^t \int_{\Omega \cap \{H > C\}} |\nabla H|^2 dx dt \leq 0. \quad (20)$$

Since $(H - C)_+ = 0$ when $t = 0$, we deduce $(H - C)_+ = 0$ for all $t \in (0, T)$ and this gives $H \leq C$ on $(0, T) \times \Omega$. But $H(u, v) \geq \frac{\lambda}{2} (u + \frac{v}{\lambda})^2$, for $u, v \geq 0$, we conclude that the solution is bounded by some constant depending on C_0 and the initial data. \square

First, Condition (14) is assumed by the following

Lemma 3.2. *Under Condition (7), the first inequality in (13) is satisfied*

Proof. The first Condition (13) is satisfied if we prove the positivity of the following quadratic form in ∇u and ∇v :

$$\begin{aligned} \Psi &= P \cdot \nabla H_u + Q \cdot \nabla H_v \\ &= (P^u H_{uu} + Q^u H_{uv}) |\nabla u|^2 + [H_{uu} P^v + (P^u + Q^v) H_{uv} + Q^u H_{vv}] \nabla u \cdot \nabla v \\ &\quad + (P^v H_{uv} + Q^v H_{vv}) |\nabla v|^2, \end{aligned} \quad (21)$$

which can be written as follows

$$\Psi = u \Psi_u + v \Psi_v + \Psi_d, \quad (22)$$

where Ψ_u , Ψ_v and Ψ_d are the following quadratic forms

$$\begin{aligned} \Psi_u &= \alpha_{11} H_{uu} |\nabla u|^2 + [b_{11} H_{uu} + (\alpha_{11} + \alpha_{21}) H_{uv}] \nabla u \cdot \nabla v + (b_{11} H_{uv} + \alpha_{21} H_{vv}) |\nabla v|^2, \\ \Psi_v &= (\alpha_{12} H_{uu} + b_{22} H_{uv}) |\nabla u|^2 + [(\alpha_{12} + \alpha_{22}) H_{uv} + b_{22} H_{vv}] \nabla u \cdot \nabla v + \alpha_{22} |H_{vv} \nabla v|^2, \\ \Psi_d &= d_1 H_{uu} |\nabla u|^2 + (d_1 + d_2) H_{uv} \nabla u \cdot \nabla v + d_2 H_{vv} |\nabla v|^2. \end{aligned} \quad (23)$$

By applying each of the above quadratic forms to the inequality

$$A |\nabla u|^2 + B \nabla u \cdot \nabla v + C |\nabla v|^2 \geq - \left(\frac{\Delta}{8C} |\nabla u|^2 + \frac{\Delta}{8A} |\nabla v|^2 \right), \quad (24)$$

for positive numbers A , B and C , with

$$\Delta = B^2 - 4AC,$$

we get

$$\Psi_u \geq - \left(\frac{|\nabla u|^2}{8C_u} + \frac{|\nabla v|^2}{8A_u} \right) \Delta_u := - \left(\frac{|\nabla u|^2}{8(b_{11} + \mu\alpha_{21})} + \frac{|\nabla v|^2}{8\alpha_{11}\lambda} \right) \Delta_u, \quad (25)$$

where

$$\Delta_u = (b_{11}\lambda - \alpha_{11} + \alpha_{21})^2 - 4\alpha_{11}\alpha_{21}(K^2 - 1). \quad (26)$$

If we choose λ such that

$$(b_{11}\lambda - \alpha_{11} + \alpha_{21})^2 - 4\alpha_{11}\alpha_{21}(K^2 - 1) < 0, \quad (27)$$

then the positivity of the quadratic form Ψ_u , amounts to the following condition

$$0 < \lambda < \frac{(\alpha_{11} - \alpha_{21}) + 2\sqrt{\alpha_{21}\alpha_{11}(K^2 - 1)}}{b_{11}}. \quad (28)$$

For the second quadratic form, we have

$$\Psi_v \geq - \left(\frac{|\nabla u|^2}{8C_v} + \frac{|\nabla v|^2}{8A_v} \right) \Delta_v := - \left(\frac{|\nabla u|^2}{8\alpha_{22}\mu} + \frac{|\nabla v|^2}{8(b_{11}\lambda + \alpha_{21})} \right) \Delta_v, \quad (29)$$

where

$$\Delta_v = [\mu b_{22} - \alpha_{22} + \alpha_{12}]^2 - 4\alpha_{12}\alpha_{22}(K^2 - 1). \quad (30)$$

By choosing μ such that

$$[\mu b_{22} - \alpha_{22} + \alpha_{12}]^2 - 4\alpha_{12}\alpha_{22}(K^2 - 1) < 0, \quad (31)$$

the quadratic form Ψ_v is positive, under the condition

$$0 < \mu < \frac{(\alpha_{22} - \alpha_{12}) + 2\sqrt{\alpha_{12}\alpha_{22}(K^2 - 1)}}{b_{22}}. \quad (32)$$

Finally, we have

$$\Psi_d \geq - \left(\frac{|\nabla u|^2}{8C_d} + \frac{|\nabla v|^2}{8A_d} \right) \Delta_d = - \left(\frac{|\nabla u|^2}{8d_2\mu} + \frac{|\nabla v|^2}{8\lambda d_1} \right) \Delta_d, \quad (33)$$

where

$$\Delta_d = (d_1 + d_2)^2 - 4K^2 d_1 d_2. \quad (34)$$

Then under Condition (7), we can find a neighborhood of $K = 1$ such that for all λ and μ satisfying (28) and (32) with $\lambda\mu = K^2$, we have the first inequality in (13). \square

Remark 3.3. We observe that, among the proof of Lemma 3.2, we can prove

$$P\nabla H_u + Q\nabla H_v \geq \lambda_1(1 + u + v) \left(|\nabla u|^2 + |\nabla v|^2 \right), \quad (35)$$

for all $(u, v) \in \Gamma \cap \{(u, v) : H(u, v) > C_0\}$.

Lemma 3.4. Under Condition (7), the second inequality in (13) is satisfied.

Proof. For the second Condition (13) which amount to the positivity of the following quadratic form in ∇u and ∇v :

$$\begin{aligned} \Phi &= (H_u \cdot P + H_v \cdot Q) \nabla H - \lambda_1(1 + u + v) |\nabla H|^2 \\ &= A_1 |\nabla u|^2 + B_1 \nabla u \cdot \nabla v + C_1 |\nabla v|^2, \end{aligned} \quad (36)$$

where

$$\begin{aligned} A_1 &= (H_u \cdot P^u + H_v \cdot Q^u) H_u - \lambda_1(1 + u + v) H_u^2, \\ B_1 &= (H_u \cdot P^u + H_v \cdot Q^u) H_v + (H_u \cdot P^v + H_v \cdot Q^v) H_u - 2\lambda_1(1 + u + v) H_u H_v, \\ C_1 &= (H_u \cdot P^v + H_v \cdot Q^v) H_v - \lambda_1(1 + u + v) H_v^2. \end{aligned} \quad (37)$$

and λ_1 any positive constant such that

$$\lambda_1 < \min \{d_i, \alpha_{ij}; i, j = 1, 2\}. \quad (38)$$

However, a simple calculation shows that its discriminant is given by

$$\begin{aligned} \Delta_1 &= [(H_u \cdot P^u + H_v \cdot Q^u) H_v - (H_u \cdot P^v + H_v \cdot Q^v) H_u]^2 \\ &= [(H_u \cdot (P^u - d_1) + H_v \cdot Q^u) H_v - (H_u \cdot P^v + H_v \cdot (Q^v - d_2)) H_u + (d_1 - d_2) H_u H_v]^2. \end{aligned} \quad (39)$$

Let us begin with the case $d_1 = d_2$: The discriminant can be written as follows

$$\Delta_1^0 = \{\alpha u^3 + \beta [(b_{11}\lambda - b_{22})u - (b_{22}\mu - b_{11})v]uv + \gamma v^3\}^2, \quad (40)$$

where

$$\begin{aligned} \alpha &= \lambda [-b_{11}\lambda + (\alpha_{11} - \alpha_{21})], \\ \beta &= (\lambda\mu - 1), \\ \gamma &= [b_{22}\mu + (\alpha_{12} - \alpha_{22})]\mu. \end{aligned} \quad (41)$$

Using (27) and (31) we can find a positive constant C_3 such that

$$\Delta_1 \leq C_3 (K^2 - 1) H^3$$

in a neighborhood of $K = 1$.

When $d_1 \neq d_2$, then the discriminant becomes

$$\begin{aligned} \Delta_1 &= [\alpha u^3 + \beta [(b_{11}\lambda - b_{22})u - (b_{22}\mu - b_{11})v]uv + \gamma v^3 + (d_1 - d_2) H_u H_v]^2 \\ &\leq 2 [\alpha u^3 + \beta [(b_{11}\lambda + b_{22})u + (b_{22}\mu + b_{11})v]uv + \gamma v^3]^2 + 2 [(d_1 - d_2) H_u H_v]^2. \end{aligned} \quad (42)$$

Since

$$\lim_{u+v \rightarrow \infty} \frac{H_u H_v}{\sqrt{\Delta_1^0}} = 0,$$

then for all $\epsilon > 0$, we can find positive constants C_4 and C_0 such that

$$\Delta_1 \leq C_4 (K^2 - 1 + \epsilon) H^3, \text{ for all } u + v > C_0 \quad (43)$$

uniformly in a bounded neighborhood of $K = 1$. Since

$$\begin{aligned} A_1 &= [(H_u \cdot P^u + H_v \cdot Q^u) - \lambda_1 (1 + u + v) H_u] H_u \\ &= [(\lambda u + v) (d_1 - \lambda_1 + (\alpha_{11} - \lambda_1) u + (\alpha_{12} - \lambda_1) v) + (u + \mu v) b_{22} v] (\lambda u + v), \\ C_1 &= (H_u \cdot P^v + H_v \cdot Q^v) H_v - \lambda_1 (1 + u + v) H_v^2 \\ &= [(\lambda u + v) b_{11} u + (u + \mu v) (d_2 - \lambda_1 + (\alpha_{21} - \lambda_1) u + (\alpha_{22} - \lambda_1) v)] (u + \mu v), \end{aligned} \quad (44)$$

are positive homogeneous polynomials in $u > 0$ and $v > 0$ of third degree and as a simple calculation shows that

$$C_5 (|\nabla u|^2 + |\nabla v|^2) H \leq |\nabla H|^2 \leq C_6 (|\nabla u|^2 + |\nabla v|^2) H, \quad (45)$$

for some positive constants C_5 and C_6 , then using inequality (24), we can find another constant C_7 and a constant C_0 (independent of K bounded) such that

$$(H_u \cdot P + H_v \cdot Q) \nabla H \geq \lambda_1 (1 + u + v) |\nabla H|^2 - C_7 (K^2 - 1 + \epsilon) (u + v) |\nabla H|^2, \quad (46)$$

for all $u + v > C_0$. Taking $\epsilon < \frac{\lambda_1}{2C_7}$ and a neighborhood of $K = 1$ in which $K^2 - 1 < \frac{\lambda_1}{2C_7}$, we get the second Condition (13). \square

Remark 3.5. We observe that, in the proof of Lemma 3.4, we can prove

$$(H_u \cdot P + H_v \cdot Q) \nabla H \geq \lambda_1 (1 + u + v) |\nabla H|^2, \quad (47)$$

for all $(u, v) \in \Gamma \cap \{(u, v) : H(u, v) > C_0\}$.

Condition (14) is assumed by the following

Lemma 3.6. *Condition (14) is satisfied.*

Proof. We have

$$H_u f + H_v g = a_1 u (\lambda u + v) + a_2 v (u + \mu v) - \Phi(u, v), \quad (48)$$

with

$$\Phi(u, v) = \lambda b_1 u^3 + (-\lambda c_1 + b_1 - b_2) u^2 v + (-c_1 + c_2 - \mu b_2) u v^2 + \mu c_2 v^3. \quad (49)$$

Since

$$\lim_{u+v \rightarrow \infty} \frac{a_1 u (\lambda u + v) + a_2 v (u + \mu v)}{\Phi(u, v)} = 0,$$

then we have (14). \square

Proof. of Theorem 2.1 Using the above lemmata, we deduce easily the inequalities (13) and (14) of Theorem 3.1 and then the boundedness of the positive solutions to problem (1)-(4) under the Condition (7). \square

Proof. of Theorem 2.2 The proof is an immediate consequence of Theorem 1.1, Theorem 2.1, and the preliminary observations. \square

4 Numerical Simulations and Discussion

In this section we would like to support our results in Section 2 by simulating (1)-(4). Our primary goal is to choose parameters that satisfy Condition (10), whilst violating (6), and then simulate our system for a host of different initial conditions. If one observes Case 1 in Table 1

$$\alpha_{11} - \alpha_{21} = 0.1 - 0.06 < 0.06 = b_{22},$$

we see Condition (iii) of (6), is violated, but our Condition (iv) of (10) holds.

If one observes Case 2 in Table 1

$$\alpha_{11} - \alpha_{21} = 1.2 - 0.3 < 1 = b_{22},$$

again we see Condition (iii) of (6), is violated, but our Condition (iv) of (10) holds.

We perform our numerical simulations in MATLAB R2013b, in two space dimensions with domain $\Omega = [0, \pi] \times [0, \pi]$. The spectral Galerkin method was used to approximate u, v defined in (1)-(2), as $u^{(n)} := \sum_{j,k=0}^n \mu_{1,j,k} \varphi_{j,k}$, $v^{(n)} := \sum_{j,k=0}^n \mu_{2,j,k} \varphi_{j,k}$, where

$$\varphi_{j,k}(x, y) = \begin{cases} \frac{1}{\pi}, & \text{if } j, k = 0 \\ \frac{\sqrt{2}}{\pi} \cos(ky), & \text{if } j = 0, k \neq 0 \\ \frac{\sqrt{2}}{\pi} \cos(jx), & \text{if } j \neq 0, k = 0 \\ \frac{2}{\pi} \cos(jx) \cos(ky), & \text{if } j, k \neq 0 \end{cases}. \quad (50)$$

To approximate $\mu_{i,j,k}$ ($i = 1, 2$), we use the methods described in [40] to express $\mu_{1,j,k}$ as the following

ordinary differential equation that can be solved numerically using the Matlab function `ode113`:

$$\begin{aligned}
\frac{d}{dt}\mu_{1,\tilde{j},\tilde{k}} &= (a_1 - (\tilde{j}^2 + \tilde{k}^2)d_1)\mu_{1,\tilde{j},\tilde{k}} \\
&\quad - (\tilde{j}^2 + \tilde{k}^2) \sum_{l,m=0}^n \sum_{\tilde{l},\tilde{m}=0}^n (\alpha_{11} + \alpha_{12} + b_{11})\mu_{1,l,m}\mu_{2,\tilde{l},\tilde{m}} \int_{\Omega} \varphi_{l,m}\varphi_{\tilde{l},\tilde{m}}\varphi_{\tilde{j},\tilde{k}} \\
&\quad + \sum_{l,m=0}^n \sum_{\tilde{l},\tilde{m}=0}^n (\alpha_{11}\mu_{1,l,m}\mu_{1,\tilde{l},\tilde{m}} + \alpha_{12}\mu_{1,l,m}\mu_{2,\tilde{l},\tilde{m}} + b_{11}\mu_{2,l,m}\mu_{1,\tilde{l},\tilde{m}}) \int_{\Omega} \varphi_{l,m}\nabla\varphi_{\tilde{l},\tilde{m}} \cdot \nabla\varphi_{\tilde{j},\tilde{k}} \\
&\quad - \sum_{l,m=0}^n \sum_{\tilde{l},\tilde{m}=0}^n (b_1\mu_{1,\tilde{l},\tilde{m}} + c_1\mu_{2,\tilde{l},\tilde{m}})\mu_{1,l,m} \int_{\Omega} \varphi_{\tilde{j},\tilde{k}}\varphi_{l,m}\varphi_{\tilde{l},\tilde{m}}
\end{aligned}$$

A similar method is used to approximate $\mu_{2,j,k}$.

We now provide the results of numerical simulations on (1)-(4). Two parameter cases are selected (see Table 1). Notice that Case 1 does indeed fit the form of the SKT model [36]. Nine simulations were run for each parameter case under various initial conditions for u and v by selecting permutations of the densities given in Figure 1a-1c. The final distributions for u and v for both cases under *all* initial conditions approached spatial homogeneity as described in [24]. This shows that for various initial conditions, our result Theorem 2.2 is verified, that is, solutions approach steady state, so exist globally, whilst the results of [20] are not applicable here.

	Case 1	Case 2
d_1	0.01	0.25
d_2	0.1	0.5
a_1	1	0.2
b_1	2	0.8
c_1	0.2	0.8
a_2	0.3	0.3
b_2	1	0.4
c_2	4	0.9
α_{11}	0.1	1.2
α_{12}	0.12	0.25
α_{21}	0.06	0.3
α_{22}	0.8	0.75
b_{11}	0.12	0.1
b_{22}	0.06	1

Table 1: Parameters used in simulations

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References

- [1] N. Alikakos, L^p -Bounds of Solutions of Reaction-Diffusion Equations. Comm.. P. D. E. 4 (1979), pp. 827-828.
- [2] H. Amman, Dynamic theory of quasilinear parabolic systems – III. Global existence. Math. Z. 202, (1989), pp. 211-250.

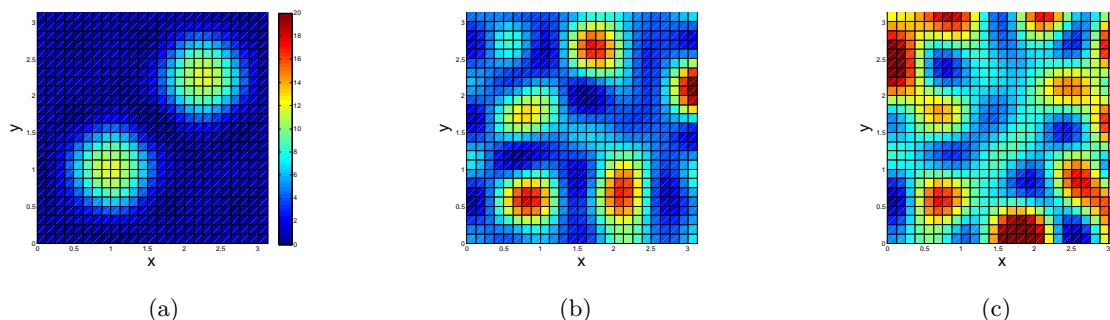


Figure 1: Figures 1a-1c represent possible initial densities of the species u and v . We run 9 different tests by selecting initial conditions for u and v from 1a, 1b, and 1c. The color scheme is the same for all plots.

- [3] S. Berres, R. Ruiz-Baier, H. Schwandt, E.M. Tory, An adaptive finite-volume method for a model of two-phase pedestrian flow. *Networks and Heterogeneous Media*, Vol. 6, number 3, (2011).
- [4] L. Chen and A. Jungel, Analysis of a parabolic cross-diffusion population model without self-diffusion. *J. Diff. Eqn.*, Vol. 224, number 11, (2006), pp. 39-59.
- [5] L. Chen and A. Jungel, Analysis of a multi-dimensional parabolic population model with strong cross-diffusion. *SIAM J. of Math. Anal.*, Vol. 36, (2004), pp. 301-322.
- [6] A. Friedman, *Partial Differential Equations of Parabolic Type*. Prentice Hall Englewood Cliffs. N. J., 1964.
- [7] G. Galiano, M. L. Garzon and A. Jungel, Semi-discretization and numerical convergence of a nonlinear cross-diffusion population model. *Numer. Math.*, Vol. 93, (2003), pp. 655-673.
- [8] D. Henry, *Geometric Theory of Semi-linear Parabolic Equations*. Lecture Notes in Mathematics 840, Springer-Verlag, New-York, 1984.
- [9] S. L. Hollis, R. H. Martin and M. Pierre, Global Existence and Boundedness in Reaction Diffusion Systems. *SIAM. J. Math. Anal.*, Vol. 18, number 3, (1987).
- [10] O. John and J. Stara, Some (new) counterexamples of parabolic systems. *Commentat. Math. Univ. Carol.* 36, (1995), pp. 503-510.
- [11] A. Jungel and N. Zamponi, Boundedness of weak solutions to cross diffusion systems from population dynamics, arXiv:1404.6054, (2014).
- [12] M. Kirane and S. Kouachi, Asymptotic Behavior for a System Describing Epidemics with Migration and Spatial Spread of Infection. *Dynamical Systems and Applications*, Vol 12, (1993), pp. 121-130.
- [13] M. Kirane and S. Kouachi, A strongly nonlinear reaction diffusion model for a deterministic diffusive epidemic. *Japan Journal of Industrial and Applied Mathematics*, Volume 12, number 1, (1995).
- [14] M. Kirane and S. Kouachi, Global Solutions to a System of Strongly Coupled Reaction-Diffusion Equations. *Nonlinear Analysis Theory, Methods and Applications*, Vol 126, (1996).
- [15] S. Kouachi, Global existence for coupled reaction diffusion systems modelling some reversible chemical reactions, *Dynamics of Partial Differential Equations*, Volume 8, Number 2, (2011), pp. 79-88.

- [16] S. Kouachi, Existence of global solutions to reaction-diffusion systems via a Lyapunov functional, *Electron. J. Differential Equations*, 68, (2001), pp. 1-10.
- [17] S. Kouachi, Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional *Electronic Journal of Differential Equations*, Vol. 2002(2002), No. 88, pp. 1-13.
- [18] S. Kouachi, Global existence for reaction diffusion systems without nonlinearities growth condition, *Mathematical Methods in the Applied Sciences*, volume 34, issue 7, (2011), pp. 798-802.
- [19] K.Kuto, Stability and Hopf bifurcation of coexistence steady-states to an SKT model in spatially heterogeneous environment, *Discrete Contin. Dynam. Sys. A*, volume 24 (2009), pp. 489-509.
- [20] D. Le, Global existence for a class of strongly coupled parabolic systems, *Annali di Matematica* 185, (2006), 133-154.
- [21] D. Le, T. Nguyen, Global existence for a class of triangular parabolic systems on domains of arbitrary dimension, *Proc. Amer. Math. Soc.* 133 (7), (2005), pp. 1985-1992.
- [22] D.Le and T.Nguyen, Everywhere regularity of solutions to a class of strongly coupled degenerate parabolic systems, *Comm. in PDE's.*, 31, no.2, (2006), pp. 307-324.
- [23] Y.Lou, S.Martinez and W.M.Ni, On 3 x 3 Lotk-Volterra competition systems with cross-diffusion. *Discrete Contin. Dynam. Sys A*, 6 (2000), pp. 175-190.
- [24] Y. Lou and M.Ni, Diffusion, self-diffusion and cross-diffusion. *J. Diff. Eqn.*, Vol.131, (1996), pp.79-131.
- [25] Y.Lou, W.M.Ni and Y.Wu On the global existence of a cross-diffusion system. *Discrete Contin. Dynam. Sys A*, 4, (1998), pp. 193-203.
- [26] K. Masuda, On the Global Existence and Asymptotic Behavior of Solutions of Reaction-Diffusion Equations. *Hokkaido. Math. J.* 12, (1983), pp. 360-370.
- [27] H. Matano and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, *Publ. Res. Inst. Math. Sci.* 19, (1983), pp. 1049-1079.
- [28] M. Mimura and K. Kawasaki, Spatial segregation in competitive interaction diffusion equations, *J. Math. Biol.* 9, (1980), pp. 49-64.
- [29] J. Morgan, Global Existence for Semilinear Parabolic Systems, *SIAM J. Math. Anal.* 20, (1989), pp. 1128-1144.
- [30] C.Pao, Strongly coupled elliptic systems and applications to Lotka-Volterra models with cross-diffusion, *Nonlin. Anal.: Theory Meth. Appl.* 60, (2005), pp. 1197-1217.
- [31] R. D. Parshad, S. Kouachi and J. B. Gutierrez, Global Existence and Asymptotic Behavior of a Model for Biological Control of Invasive Species via Supermale Introduction, *Commun. Math. Sci.*, Vol. 11, No. 4, (2013), pp. 971-992.
- [32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Math. Sciences 44, Springer-Verlag, New York, 1983.
- [33] F. Rothe, *Global Solutions of Reaction-Diffusion Systems*, Lecture Notes in Math. 1072, Springer-Verlag, Berlin, 1984.
- [34] K. Ryu and I. Ahn, Positive steady-states for two interacting species models with linear self-cross-diffusions. *Discrete Contin. Dynam. Sys. A* 9, (2003), pp. 1049-1061.
- [35] K. Ryu and I. Ahn, Coexistence states of certain population models with nonlinear diffusions among multi-species. *Dynam. Contin. Discrete Impuls. Syst. Ser A:Math Anal.* 12, (2005), pp. 235-246.

- [36] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theor. Biol.*, 79(1), (1979), pp. 83-99.
- [37] S. Shim, Long time properties of prey-predator system with cross diffusion, *Comm. Korean Math. Soc* 21, (2006), pp. 293-320.
- [38] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.
- [39] S. Xu, Existence of global solutions for a predator prey model with cross diffusion, *Electron. J. Differential Equations*, Vol 2008 (2008), No. 06, pp. 1-14.
- [40] K.E. Yong. *A mathematical model of the interactions between pollinators and their effects on pollination of almonds*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)—The University of Iowa.
- [41] K.E. Yong, Yi Li, S.D. Hendrix, Habitat choice of multiple pollinators in almond trees and its potential effect on pollen movement and productivity: A theoretical approach using the Shigesada–Kawasaki–Teramoto model, *Journal of Theoretical Biology* 305, (2012), pp. 103-109.